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Series studies of the Potts model: II. Bulk series for the square lattice

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Abstract. The finite-lattice method of series expansion has been used to extend low-temperature series for the partition function, order parameter and susceptibility of the q-state Potts model to order z^{56} (i.e. u^{28}), z^{47}, z^{43}, z^{39}, z^{39}, z^{39}, z^{35}, z^{31} and z^{31} for $q = 2, 3, 4, \ldots, 9$ and 10 respectively. These series are used to test techniques designed to distinguish first-order transitions from continuous transitions. New numerical values are also obtained for the q-state Potts model with $q > 4$.

1. Introduction

This is the second in a series of papers in which we study the critical behaviour of the q-state Potts model in both two and three dimensions using series expansions derived from the finite-lattice method. The previous paper (Guttmann and Enting 1993), denoted I hereafter, gave the general expressions used to derive high- and low-temperature expansions for the q-state Potts model. In I, series expansions for the $q = 2$ (Ising) case on the simple cubic lattice were analysed. The present paper derives and analyses series for the bulk thermodynamic properties for Potts model on the square lattice for integer q ranging from 2 to 10.

A brief history of the model follows. After the initial paper by Potts (1952), the model attracted little attention for almost two decades. During the 1970s there was greatly renewed interest, with new exact results, series studies and renormalization group calculations as well as applications to phase transitions in surface films. A particular concern at that time was the failure of renormalization group calculations to reproduce the exact results for the order of the transition in two dimensions. A review by Wu (1982) described much of the work on the Potts model.

The main exact results come from Potts (1952) and Baxter (1973, 1982). In particular, Potts (1952) located the critical temperature exactly for the two-dimensional Potts model on a square lattice by duality arguments. He found that $T_c = \Delta E/(k \ln(1 + \sqrt{q}))$ and $\bar{U} = \frac{1}{2}(U_c^+ + U_c^-) = \Delta E(1 - 1/\sqrt{q})$. (The energy is relative to the ground-state energy; other aspects of the notation are defined in section 2 below.)

For $q > 4$ the model has a first-order phase transition, and Baxter (1973) obtained the free energy and latent heat in 1973, and subsequently the value of the magnetization at T_c (Baxter 1982). The results are listed in tables 1 and 2 with the free energies converted to be consistent with our choice of zero ground-state energy. The free energy, f, defined

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by Baxter (1973) is related to the free energy, F, which we define below through the dimensionless form $F/\Delta E = f/\Delta E + 2$. Further, Baxter showed that the values of the magnetization at T_c were the same for the square, triangular and honeycomb lattices—a consequence of the star–triangle relation. For $q \leqslant 4$ the model has a second-order phase transition. Indeed, for $q = 2$ the model is just the usual spin-$\frac{1}{2}$ Ising model, while $q = 4$ is the 'marginal' q value, at which the exponents have confluent logarithmic corrections.

Table 1. Exact critical properties for $q \leqslant 4$, from the work of Potts (1952) and Baxter (1973). The internal energies U and free energy F are defined relative to a ground-state energy of zero. Numerical estimates of $F_c/\Delta E$ are also shown.

q	z_c	$U_c/\Delta E$	$F_c/\Delta E$	$F_c/\Delta E$(series)
2	0.414 214	0.292 893	−0.054 826	−0.054 825
3	0.366 035	0.422 650	−0.059 780	−0.059 777
4	0.333 333	0.500 000	−0.056 708	−0.056 722

Table 2. Exact properties at the transition point for $q \geqslant 5$, from the work of Potts (1952) and Baxter (1973, 1982). The energies and free energy are defined relative to a ground-state energy of zero.

q	$kT_c/\Delta E$	ΔM	$\Delta U/\Delta E$	$\bar{U}/\Delta E$	$F_c/\Delta E$
5	0.851 528	0.492 141	0.052 919	0.552 786	−0.052 05
6	0.807 607	0.665 181	0.201 464	0.591 752	−0.047 38
7	0.773 059	0.749 565	0.353 277	0.622 036	−0.043 11
8	0.744 904	0.799 837	0.486 358	0.646 447	−0.039 35
9	0.721 348	0.833 261	0.599 668	0.666 667	−0.036 08
10	0.701 232	0.857 107	0.696 049	0.683 772	−0.033 23

Table 3. Exact critical exponents for two-dimensional Potts models.

q	α	β	γ	δ	ν	η	Δ_1
2	0	1/8	7/4	15	1	1/4	4/3
3	1/3	1/9	13/9	14	5/6	4/15	2/3
4	2/3	1/12	7/6	15	2/3	1/4	0

The critical exponents and critical temperature for $q \leqslant 4$ are shown in table 3. The thermal exponent was given by Black and Emery (1981), following a conjecture of den Nijs (1979). Den Nijs (1983) also obtained the magnetic exponent, while Nienhuis (1982) obtained the (thermal) correction-to-scaling exponent Δ_1. The results are:

$$2 - \alpha = \frac{2}{y_t} = \frac{2(2-r)}{3(1-r)}$$

$$1 + \frac{1}{\delta} = \frac{2}{y_h} = \frac{8(2-r)}{(3-r)(5-r)}$$

$$\Delta_1 = \frac{4r}{3(1-r)}$$

with

$$0 \leqslant r \equiv \frac{2}{\pi}\cos^{-1}(\sqrt{q}/2) \leqslant 1 \qquad \text{for } 0 \leqslant q \leqslant 4.$$

However, for $q > 4$ certain properties still remain unknown. These include the value of the specific heat and of the isothermal susceptibility at the critical temperature. Various surface critical exponents and critical values are also unknown.

Of even greater interest is the behaviour of the three-dimensional Potts model. As noted above, for the $q = 2$ (Ising) case, the low-temperature series and some high-temperature series have recently been extended in I. For $q = 3$, the three-dimensional Potts model is of particular interest as it constrains the order of the deconfinement transition in quantum chromodynamics. The key question is whether the $q = 3$, $d = 3$ Potts model transition is first or second order. This is discussed further in the third of our series of papers.

The fact that the critical behaviour is known in the two-dimensional case makes it an ideal 'test-bed' for methods to distinguish first-order from second-order phase transitions. In this paper we have extended the low-temperature (and by duality, the zero-field high-temperature) series for the two-dimensional model for $q = 2, 3, 4, \ldots, 9, 10$. By the use of the finite-lattice method (see I and references therein), quite substantial series extensions have been made. By using differential approximants (Guttmann 1989) to integrate the series, we have been able to clearly distinguish between first-order and second-order phase transitions.

The layout of the remainder of the paper is as follows. In the next section we briefly describe the finite-lattice method and the nature of the results we thus obtained. In section 3 we analyse the data. In section 4 we present a discussion of the results.

2. Series expansions from the finite-lattice method

The definitions and notation follow the usage of I. The standard q-state Potts model is defined on a lattice with each site having a 'spin' variable that takes on q possible values (denoted '0' to $q - 1$). An energy $\Delta E > 0$ is associated with each pair of interacting sites that are in different spin states, and an energy of 0 applies to pairs of interacting sites in the same state. We consider only the square lattice, with each site interacting only with its four nearest neighbours. Each site not in state '0' has an additional field energy H.

The thermodynamic quantities can be derived from the partition function Z. We choose the normalization such that the state with all sites in state '0' has zero energy. This particular normalization of the partition function is commonly denoted Λ.

We work in terms of the expansion variables $z = \exp(-\Delta E/kT)$, $\mu = \exp(-H/kT)$ and the high-temperature variable $v = (1 - z)/(1 + (q - 1)z)$.

For the square lattice, the high-temperature expansion for the partition function takes the form (see I for the general case):

$$\Lambda = q^{-1}(1 + (q - 1)z)^2\Phi(v) = q(1 + (q - 1)v)^{-2}\Phi(v) \tag{1}$$

with

$$\Phi(v) = 1 + (q - 1)v^4 + \ldots . \tag{2}$$

For the low-temperature expansion, we use a modified field variable $x = 1 - \mu$ and truncate at order x^2 so that the partition function is expressed as

$$\Lambda = \Lambda_0 + x\Lambda_1 + x^2\Lambda_2 + \ldots . \tag{3}$$

The zero-field partition function is expanded as

$$\Lambda_0(z) = \sum_n \lambda_n z^n. \tag{4}$$

On the square lattice, the duality relation takes the form

$$\Lambda_0(x) = \Phi(x) \qquad \textit{(square lattice only)} \tag{5a}$$

so that the transition point (assuming it is unique) occurs (Potts 1952) at

$$z_c = v_c = \frac{1}{1+\sqrt{q}} \qquad \text{(square lattice only).} \tag{5b}$$

The dimensionless free energy is given by

$$\frac{F}{\Delta E} = -\frac{kT}{\Delta E} \ln \Lambda \tag{6}$$

so that for low temperatures, the internal energy is given by

$$U = \Delta E\, z \frac{\mathrm{d}\Lambda_0}{\mathrm{d}z} / \Lambda_0 \tag{7}$$

the order parameter by

$$M = 1 + \frac{q}{q-1}\frac{\Lambda_1}{\Lambda_0} = \sum_n m_n z^n \tag{8}$$

and the susceptibility by

$$\chi = 2\frac{\Lambda_2}{\Lambda_0} - \frac{\Lambda_1}{\Lambda_0} - \left(\frac{\Lambda_1}{\Lambda_0}\right)^2 = \sum_n c_n z^n. \tag{9}$$

Note that in I the expansions were expressed in terms of $u = z^2$, as only even powers of z occur for $q = 2$. For $q \geqslant 3$ an additional 'transverse' susceptibility can be defined (Straley and Fisher 1973) but is not considered here.

For $T \geqslant T_c$ the internal energy is given by

$$U = \Delta E \frac{v}{2}\frac{q-1}{q}(1-v) - \Delta E \frac{(1-v)(1+(q-1)v)}{q}\frac{\mathrm{d}}{\mathrm{d}v}\ln \Phi(v) \tag{10}$$

where v is the lattice coordination number (4 in this case). Series expansions for the Potts model on the square lattice had been obtained previously by a number of workers. Kihara *et al* (1954) obtained the general-q zero-field free-energy series to order z^{16} (or equivalently to v^{16} by virtue of duality). Straley and Fisher (1973) obtained the general-q general-field low-temperature series to z^{13}. Enting (1974) analysed the field grouping for $q = 3$ to order μ^9, however, the series were not published directly, but rather as 'coded' partial generating functions, based on the formalism of Sykes *et al* (1965).

The 3-state square lattice Potts model was the first application of the finite-lattice method (de Neef 1975, de Neef and Enting 1977). This work obtained the high-temperature expansion to order 23 (working in powers of $\Delta E/kT$). The finite-lattice method was used

subsequently to obtain low-temperature expansions for Λ and M for $q = 3$ to order z^{31} (Enting 1980). These series were extended to z^{35} (and new series for χ added) by Adler *et al* (1982). The algorithm used in the present work is essentially unchanged from that used in the 1980 and 1982 studies, although the program has been modified to work with general integer values of q. The increased number of terms that we have obtained reflects the increase in available computing capacity over the last decade rather than any major change to our technique. Recently, a preprint by Bhanot *et al* (1993) gives series for $q = 3$ and $q = 8$ on the square lattice (and some new $d = 3$ series). They used a transfer-matrix method related to our approach but with a more complicated (and apparently less efficient) choice of boundary conditions for their finite-lattices. Their square lattice series extend earlier results but are slightly shorter than those presented here.

The basic formulation of the finite-lattice method approximates the partition function per site, Z, as

$$Z = \lim_{|\Gamma|\to\infty} Z_\Gamma^{1/|\Gamma|} \approx \prod_{\alpha\in A} Z_\alpha^{W(\alpha)} \tag{11a}$$

where Γ denotes a graph (with $|\Gamma|$ sites) which is allowed to become arbitrarily large and A is a set of finite-lattices, α, with A closed under the operation of intersection of finite-lattices. For the square lattice, this general relation has the specific form:

$$Z = \lim_{N\to\infty} Z_{NN}^{1/N^2} \cong \prod_{[q,r]\in A} Z_{qr}^{W(q,r)} \tag{11b}$$

where Z_{qr} is the partition function of a rectangle of dimensions $q \times r$ sites. For low-temperature expansions, the Z_{qr} are to be evaluated with a surrounding layer of fully ordered sites. The weights $W(q, r)$ depend on the set, A, over which the product is taken. In approximations (11*a*),(11*b*) an appropriate choice of weights will give Z as a series correct up to, but not including, the order of the first connected graph that will not fit into any of the rectangles of set A (Enting 1978a).

For low-temperature Potts model series, the appropriate finite-lattices are rectangles of $q \times r$ sites, surrounded by a boundary of sites fixed in state '0'. Inspection of the low-temperature expansion of the Potts model shows that the limiting graphs are trees that do not double back in any direction: all lines drawn perpendicular to bonds of the lattice intersect such trees at most once. Such a tree can span a rectangle of size $q \times r$ with $q + r - 1$ sites and $q + r - 2$ bonds in the tree and will give powers of $2(q + r)$ or more in the Potts model low-temperature variable, z. If one includes all rectangles such that $q + r \leqslant k$ then the series are correct to z^{2k+1}. We denote the set of rectangles with $q + r \leqslant k$ by $A(k)$.

The combinatorial factors from Enting (1978b) give

$$W(w, \ell) = \sum_{[q,r]\in A(k)} \eta(q - w)\,\eta(r - \ell) \qquad \text{for } [w, \ell] \in A(k) \tag{12}$$

where

$$\eta(0) = 1 \tag{13a}$$

$$\eta(1) = -2 \tag{13b}$$

$$\eta(2) = 1 \tag{13c}$$

$$\eta(k) = 0 \qquad \text{otherwise.} \tag{13d}$$

This implies

$$W(w,\ell) = 1 \qquad \text{for } w+\ell = k \tag{14a}$$
$$= -3 \qquad \text{for } w+\ell = k-1 \tag{14b}$$
$$= 3 \qquad \text{for } w+\ell = k-2 \tag{14c}$$
$$= -1 \qquad \text{for } w+\ell = k-3 \tag{14d}$$
$$= 0 \qquad \text{otherwise.} \tag{14e}$$

In actual computation it is convenient to exploit the symmetry and consider only $w \leqslant \ell$. We define $B(k) = \{[q,r] : q+r \leqslant k, q \leqslant r\}$. The expansion becomes

$$Z \approx \prod_{[q,r]\in B(k)} Z_{qr}^{V(q,r)} \tag{15}$$

with

$$V(\ell,w) = 2W(\ell,w) \qquad \text{for } w < \ell \tag{16a}$$
$$V(w,w) = W(w,w) \tag{16b}$$
$$V(\ell,w) = 0 \qquad \text{for } w > \ell. \tag{16c}$$

Therefore we choose a maximum width $w_{\max}$ and work with $\ell + w \leqslant 2w_{\max}+1$. This gives series correct to $z^{4w_{\max}+3}$.

The partition functions are constructed by using a transfer-matrix formalism to build up ℓ columns of length w. As in all of the most recent applications of the finite-lattice method, we used the approach of building up the finite-lattices one site at a time. The computational complexity of the calculation is determined by the largest value of w that is required.

Storage is required for vectors giving the partial generating functions for all possible configurations of sites across a lattice. Without any simplification, such a vector will have q^w elements for a rectangle of width w. Each element must have sufficient storage for a series truncated at the requisite order (i.e. $4(k+1)$ temperature terms times three field terms in the present case). Building up the lattice one site at a time means that the 'transfer matrix' is extremely sparse and the non-zero elements of the matrix can be calculated as required rather than having to be stored. The energies defined above only single out the '0' state and so the equivalence of the other states can be used to reduce the size of the vectors to approximately $q^w/(q-1)!$.

The precise size, $R(w,q)$ of vectors required to treat a lattice w sites across is given in terms of r_{wm} the number of ways of colouring w sites with colours 0 to $q-1$, treating all permutations of colours 1 to $q-1$ as equivalent. That is

$$R(w,q) = \sum_{m=1}^{q} r_{wm} \tag{17a}$$

with $r_{01} = r_{j1} = 1$ and the general relation

$$r_{jm} = r_{j-1,m} + m r_{j-1,m}. \tag{17b}$$

The series coefficients λ_n, m_n and c_n for $q = 2$ to $q = 10$ are listed in tables A1–A9 in the appendix. For $q = 2$ the coefficients λ_n and m_n are known from the exact solutions.

The series c_n for $q = 2$ corrects two minor errors in the last two coefficients c_{44} and c_{46} obtained by Baxter and Enting (1979) using the corner-transfer-matrix technique. (The error in the work of Baxter and Enting arose from insufficient precision in the summation of their high-field polynomials—a full-precision summation of their published coefficients for the expansion (in powers of $u = z^2$ and μ) gives the results obtained here.)

Although the computer program used here is restricted to integer $q \geqslant 2$, the Potts model can be generalized to non-integer q (Fortuin and Kasteleyn 1972). A number of interesting special cases occur—in particular the limit $q \to 1$ (on any lattice) gives the statistics of the bond percolation problem (Fortuin and Kasteleyn 1972, Wu 1978). This connection has been exploited in series derivations (Enting 1986). The finite-lattice method is applicable to general q and indeed one of the earliest applications of the method was in calculating the limit of chromatic polynomials which correspond to the $T \to 0$ limit of the antiferromagnetic Potts model, expressed as a function of q (Kim and Enting 1979).

3. Analysis of series

For a second-order phase transition, quantities such as the order parameter vanish at T_c, while quantities such as the susceptibility and specific heat diverge to infinity. For a first-order phase transition, all these quantities are expected to attain a finite, non-zero value at T_c, with finite slope at T_c. However, little is known rigorously about the nature of the transition. Among the possibilities are: (i) Finite specific heats and susceptibilities allowing analytic continuation of the thermodynamic quantities beyond the transition point into a metastable region with a singularity $T_c^* > T_c$ on a 'pseudo-spinodal' line and effective 'critical exponents' at T_c^*. (ii) Finite specific heats and susceptibilities with a weak, essential singularity at T_c. Even with an essential singularity it may be possible to define the thermodynamic functions in the metastable region by analytic continuation in the complex plane passing *around* the singularity. (iii) Divergences in specific heats and susceptibilities (or their derivatives) at T_c. A previous attempt to use series expansions to search for an essential singularity gave inconclusive results (Enting and Baxter 1980). Kim and Joseph (1975) presented evidence of 'possible diverging fluctuations at the first-order transition'. However, none of our results seem to indicate any sort of singularity at the first-order transitions.

In analysing series expansions around the origin by Dlog Padé approximants or, more generally, differential approximants, poles and residues of the approximants will provide estimators of T_c and the critical exponent in the case of a second-order phase transition, while in the case of a first-order transition, the approximant will furnish an effective analytic continuation, and provide estimators of T_c^* and some effective exponent. If T_c is exactly known, as it is for the two-dimensional Potts models, this observation provides an effective means to distinguish between the two types of phase transitions. We show this in table 4, where we give the Dlog Padé approximants to the magnetization series for the $q = 3$ Potts model, the $q = 5$ Potts model and the $q = 10$ Potts model. These are representative of a second-order transition, a weak first-order transition, and a first-order transition, respectively. For the $q = 3$ case, we find $T_c^*/T_c = 0.999\,90$ and $\beta \approx 0.109$, compared to the exact result $\beta = \frac{1}{9}$, so that the apparent critical temperature is less than 0.01% *below* the true critical temperature, while the critical exponent is correct to the quoted accuracy (the true value is $\frac{1}{9}$ exactly). For the $q = 5$ Potts model, we find $T_c^*/T_c = 1.000\,64$ and $\beta \approx 0.077$, so that the apparent critical temperature is more than 0.06% *above* the true critical temperature, while the 'critical exponent' is rather erratically estimated as ≈ 0.077 . For the $q = 10$ Potts

Table 4. Singularities estimated from Padé approximants to spontaneous magnetization of the Potts model for q =3, 5 and 10.

N	$[N-1/N]$		$[N/N]$		$[N+1/N]$	
(*a*) 3-state Potts model spontaneous magnetization: $z_c = 0.366025\ldots$, $\beta = 0.1111\ldots$						
17	0.365 95	(0.1078)	0.366 15	(0.1155)	0.365 91	(0.1074)*
18	0.365 93	(0.1076)*	0.365 83	(0.1074)*	0.365 94	(0.1078)
19	0.365 98	(0.1084)	0.365 99	(0.1086)	0.365 99	(0.1086)
20	0.365 99	(0.1086)	0.365 99	(0.1086)	0.365 99	(0.1086)
21	0.365 99	(0.1086)				
(*b*) 5-state Potts model spontaneous magnetization: $z_c = 0.309016\ldots$						
15	0.309 19	(0.0768)	0.309 15	(0.0763)*	0.309 17	(0.0766)*
16	0.309 21	(0.0772)	0.309 21	(0.0772)	0.309 22	(0.0772)
17	0.309 21	(0.0772)*	0.309 21	(0.0772)*	0.309 22	(0.0772)*
18	0.309 21	(0.0772)*	0.309 21	(0.0772)*	0.309 20	(0.0769)*
19	0.309 21	(0.0772)*				
(*c*) 10-state Potts model spontaneous magnetization: $z_c = 0.240253\ldots$						
11	0.242 80	(0.0436)	0.241 24	(0.0346)*	0.242 88	(0.0441)*
12	0.242 88	(0.0441)*	0.242 78	(0.0433)*	0.242 69	(0.0428)*
13	0.242 69	(0.0428)*	0.242 71	(0.0430)*	0.242 67	(0.0427)*
14	0.242 68	(0.0427)*	0.242 75	(0.0432)*	0.242 76	(0.0433)*
15	0.242 76	(0.0433)*				

model, we find $T_c^*/T_c = 1.0104$ and $\beta \approx 0.047$, so that the apparent critical temperature is more than 1.0% *above* the true critical temperature, while the 'critical exponent' is rather erratically estimated as ≈ 0.05. (We emphasize that this so-called 'critical exponent' has *no physical meaning.*)

This method of analysis alone appears to provide a reliable indicator of the order of a phase transition when the critical temperature is known exactly. For a exactly second-order transition, the estimates of the critical temperature lie very slightly below T_c. (The $q = 2$ or Ising case for which the magnetization can be represented exactly by low-order Dlog Padé approximants is an exception.) Even for the marginal case of $q = 4$ (the critical dimension, where the model undergoes a second-order phase transition, but with logarithmic corrections to the critical exponents), we find $T_c^*/T_c = 0.999\,75$, and $\beta = 0.0906$, which is satisfyingly close to the exact value of $\frac{1}{11}$. For the weak first-order $q = 5$ case it is already clear that T_c^*/T_c is significantly bigger than 1, while the 'exponent' estimates are much more erratic than for the $q \leqslant 4$ case.

The other numerical approach to distinguish between a first- and second-order phase transition is to compare numerical approximations to the free energy and internal energy in the high- and low-temperature regimes. (On the square lattice, we obtain the high-temperature series by duality from the low-temperature series.) In figure 1 we show the plots of the free energy for $q = 3, 5$ and 10. For $q = 3$ the curve appears smooth, with no gradient discontinuities. At $q = 5$, a discontinuity in the gradient at T_c is already apparent, while at $q = 10$ the discontinuity in the gradient is manifest. In figure 2 we show the corresponding curves for the internal energy. The non-zero latent heat characteristic of a first-order transition is already manifest at $q = 5$. The numerical approximations used in these figures are the approximants formed by the method of differential approximants (DA) (Guttmann 1989, pp 83ff). This method generalizes Padé approximants by fitting an ordinary differential equation of the form

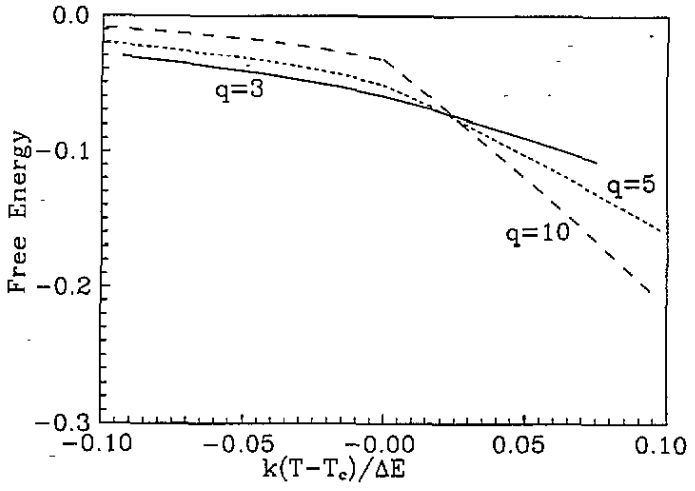

Figure 1. *Dimensionless free energy, $F/\Delta E$, of the square lattice Potts model for $q = 3$ (solid curve), $q = 5$ (short dashes) and $q = 10$ (long dashes) illustrating the change from continuous to first-order transitions. Curves are the average of typically 10 differential approximants.*

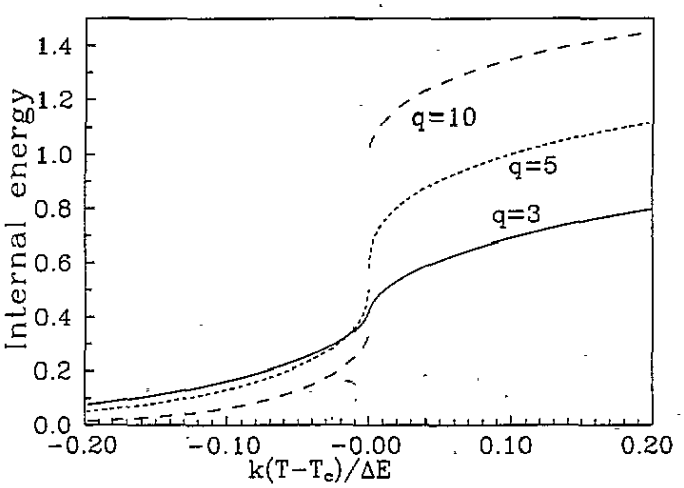

Figure 2. Dimensionless internal energy, $U/\Delta E$, of the square lattice Potts model for $q = 3$ (solid curve), $q = 5$ (short dashes) and $q = 10$ (long dashes). Each curve is the average of typically 10 differential approximants.

$$\sum_{i=0}^{m} Q_i(x) D^i f(x) = P(x)$$

(where $D^i = \mathrm{d}^i/\mathrm{d}x^i$) to the available series terms. Here $Q_k(x) = \sum_{i=0}^{m_k} q_{ki} x^i$ and $P(x) = \sum_{i=0}^{m_0} p_i x^i$ are polynomials. We chose $q_{m0} = 1$, so that the origin is not a regular singular point. This allows numerical integration of the differential equation starting at $x = 0$ in order to obtain the values plotted in the figures. For magnetization series, homogeneous DA's ($P \equiv 0$) are often most useful. (For $m = 1$ this corresponds to logarithmic derivative Padé approximants.) The degrees of Q_k and P are chosen to use all (or most) of the

available series terms. In principle, any order of differential equations can be used, but first order ($m = 1$) was mostly used in the current work. Finding the coefficients of Q_k and P reduces to solving a system of linear equations, but this system is often ill-conditioned, so that care must be taken in its solution.

This differential equation is then integrated numerically to obtain estimates of the desired physical quantities. In all cases a number (typically 10) of DA's using all the available coefficients was integrated. These were then averaged to obtain the means and standard deviations shown in the tables and graphs below. All calculations were performed in quadruple precision (approximately 34 decimal places), so that all series terms could be represented without loss of precision.

We performed the numerical integration with an extrapolation method of the Bulirsch–Stoer type, as described by Hairer (1987, section II.9). The integrations were performed in terms of the series expansion variable, but results are expressed in terms of $kT/\Delta E$.

The integrations described above clearly allow us to *qualitatively* distinguish between a first- and second-order phase transition. A much more stringent test is to *quantitatively* reproduce the magnetization gap ΔM and the latent heat ΔU for $q \geqslant 5$. These have been calculated by Baxter (1973, 1982). These exact results are shown in table 2. The agreement between numerical and exact results is, as might be expected, best for q close to 10, and worst near $q = 4$. Further details of the calculations are given in the subsections below.

We now discuss our numerical results in greater detail.

3.1. Free energy and internal energy

We integrated the equations defining approximants to the free energy and the internal energy series from $T = 0$ and $T = \infty$ to T_c in order to determine the critical value of the free-energy and the latent heat (for $q > 4$) from the discontinuity at T_c. For low temperatures, F was derived from approximants to $\ln \Lambda$ in powers of z, while for high temperatures we use approximants to $F/kT - 2\Delta E/kT$ expanded in powers of v. For the energy, high-temperature and low-temperature approximants were constructed from the respective expansions (in powers of v and z) to $U/\Delta E - 2$. For $q \leqslant 4$, table 1 compares the exact values of $F/\Delta E$ to the series estimates. For $q \geqslant 5$ table 5 gives the series estimates for $F_c/\Delta E$ and $\Delta U/\Delta E$, which should be compared to the exact values in table 2. The critical value of the free energy is obtained very accurately for all values of q. As expected from the known exact results (Baxter 1973), U was found to be continuous at T_c for $q \leqslant 4$. From the tables, it can be seen that the exact results for the latent heat are reproduced to within a few percent, except at $q = 5$. However, at $q = 5$ the latent heat is found to be larger than the exact value, which makes the order of the phase transition *more* obvious.

Table 5. Square lattice Potts model, ΔM, ΔU and F_c results from series analysis. Compare to exact values in table 2. Free energies are defined relative to a ground-state energy of zero.

q	$kT_c/\Delta E$	ΔM	$\Delta U/\Delta E$	$F_c/\Delta E$
5	0.851 528 41	0.643 ± 0.002	0.085 ± 0.015	−0.052 04
6	0.807 606 82	0.728 ± 0.010	0.200 ± 0.010	−0.047 58
7	0.773 058 89	0.775 ± 0.010	0.375 ● 0.027	−0.043 09
8	0.744 904 46	0.817 ± 0.003	0.499 ± 0.007	−0.039 51
9	0.721 347 51	0.846 ± 0.002	0.592 ± 0.014	−0.035 98
10	0.701 231 60	0.862 ± 0.030	0.706 ± 0.005	−0.033 20

The behaviour of the free energy and internal energy as functions of temperature is shown in figures 1, 2 and 3. Since our primary objective is to test techniques that are applicable in three dimensions where T_c is unknown, figure 3 which shows details of the transition region for $q = 5$ is of particular interest. The figure shows individual approximants, indicating the spread that occurs in the free energy itself and the precision with which the transition could be located if T_c were not known.

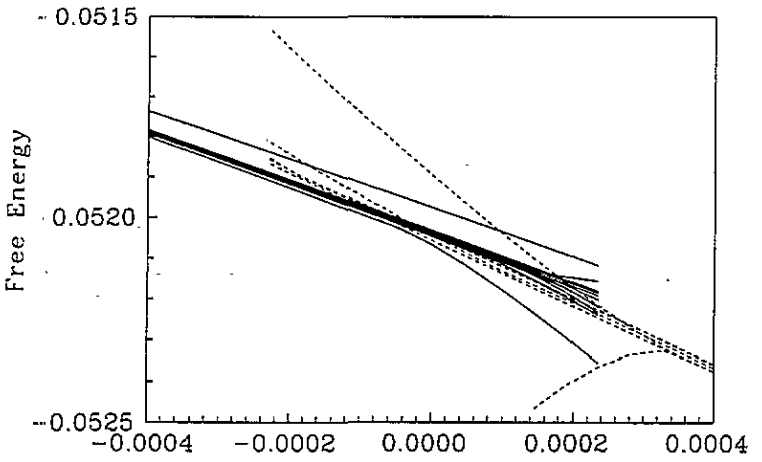

Figure 3. Detail of the approximants to the Potts model free energy ($F/\Delta E$) around the transition for $q = 5$. The plot includes seven high-temperature approximants (shown as dashed) and 13 low-temperature approximants (shown as solid), although not all of these can be distinguished on this scale.

3.2. Magnetization

We integrated the approximants to the magnetization series $M(z)$ from $T = 0$ to $T = T_c$ in order to determine the discontinuity ΔM at T_c. As expected from the known exact results (Baxter 1982), M vanished at T_c for $q \leqslant 4$. Comparing tables 2 and 5, it can be seen that the exact results are reproduced to within a few percent, except at $q = 5$. At $q = 5$ the error in the magnetization discontinuity is some 27%, falling to less than 1.5% at $q = 10$. A plot of the magnetization for several values of q is shown in figure 4.

3.3. Susceptibility and specific heat

The susceptibility and specific heat properties are not known in general. Certainly for $q \leqslant 4$ they are known to diverge at T_c, but for $q \geqslant 5$ the behaviour is less well understood. Nevertheless, it is expected that the susceptibility and specific heat should remain finite at T_c, though this has not been proved. In order to study these rapidly increasing quantities, various sequence transformation were used to generate the most appropriate series, and hence DA, for numerical integration. In general, if a quantity $f(x)$ behaves at the origin of integration like x^k, it is usually desirable to remove this term and study $f(x)/x^k$ instead. In addition, if a function is increasing rapidly, but not necessarily diverging, studying the reciprocal of the function frequently provides better converged approximants. These two

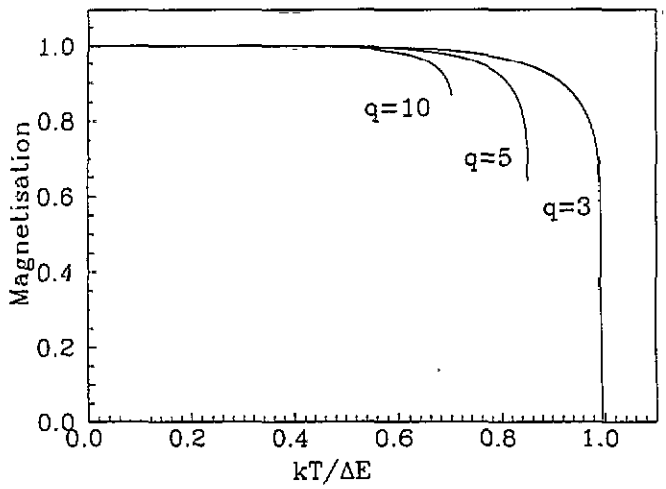

Figure 4. Spontaneous magnetization of the square lattice Potts model for $q = 3$, 5 and 10. Each curve is the average of typically 10 differential approximants.

transformations are often used together so that, for example, we worked with the series for $z^4/\chi(z)$ (where χ is the susceptibility), rather than $\chi(z)$ itself.

In table 6 we show the results for the specific heat and susceptibility at the critical temperature, when approached from both the high- and low-temperature side. We observe a monotonic decrease in the value of the specific heat at T_c with increasing q. For $q = 5$ and $q = 6$ the integration is too unreliable to quote a result. The numerical evidence gives no suggestion of asymmetry in the specific heat values above and below T_c, though our error bars are too large to give a useful test of symmetry.

Table 6. Square lattice Potts model, susceptibility and specific heat results from series analysis.

q	$kT_c/\Delta E$	$\chi(T_c^-)$	$C_0(T_c^-)$	$C_0(T_c^+)$
5	0.851 528 41	100 ± 50	—	—
6	0.807 606 82	36 ± 10	—	—
7	0.773 058 89	15 ± 4	93 ± 3	250 ± 160
8	0.744 904 46	7.3 ± 0.6	72 ± 20	75 ± 10
9	0.721 347 51	3.4 ± 0.9	40 ± 5	42 ± 4
10	0.701 231 60	2.44 ± 0.09	31.8 ± 2.8	33 ± 3

In connection with the error bars, the value for $C_0(T_c^-)$ for $q = 7$ seems anomalously low and yet, as for all the other cases, the range reflects the spread of approximants that were fitted. This anomaly serves to emphasize the fact that the ranges are obtained empirically, rather than being based on any statistical theory, and in addition reflect a relatively small number of cases.

Recently Billoire *et al* (1992) published a Monte Carlo study of the $q = 10$ Potts model specific heat. They obtained a value of 12.3 for $C_0(T_c)$ compared to our estimate of 32. However, re-analysis (Billoire, private communication) indicates that the published value is a serious underestimate, due to the fact that his system was not large enough to eliminate finite size effects. However, Billoire is able to estimate the difference between the specific heats on the ordered and disordered side of the critical temperature and finds a value of

0.447 at $q = 10$ and 0.223 at $q = 7$. Such small differences, if they exist, cannot be resolved by our numerical estimates.

For the susceptibility we also observe the same monotonic trend of decreasing values at T_c with increasing values of q. As far as we are aware, this is the first study of this quantity.

4. Discussion of results

We have shown how the finite-lattice method may be used to extend Potts model series, and have used the method to substantially extend a number of series for a range of q values. The series could all be extended by several further terms (typically four) without excessive demand on computing resources, but we did not consider this necessary for our purposes. We have used the series, combined with appropriate numerical techniques, to show how a first-order phase transition can be distinguished from a second-order transition. In this way we find extremely strong evidence for the known first-order transition for $q \geqslant 5$. The methods developed in this paper are used in a subsequent paper to investigate the nature of the phase transition for the three-state three-dimensional Potts model.

Acknowledgments

Financial support from the Australian Research Council is acknowledged. IGE wishes to thank Doochul Kim for pointing out the inconsistency in the series published by Baxter and Enting. The authors wish to thank John O'Brien, Robert Bursill and Debbie Wood for assistance with some of the numerical work.

Appendix. The series

Tables A1 to A9 list the series expansions that we have calculated.

Table A1. $q = 2$ square lattice Potts model.

	Partition function	Magnetization	Susceptibility
0	1	1	0
2	0	0	0
4	1	−2	1
6	2	−8	8
8	5	−34	60
10	14	−152	416
12	44	−714	2791
14	152	−3472	18296
16	566	−17318	118016
18	2234	−88048	752008
20	9228	−454378	4746341
22	39520	−2373048	29727472
24	174271	−12515634	185016612
26	787246	−66551016	1145415208
28	3628992	−356345666	7059265827
30	17019374	−1919453984	43338407712
32	81011889	−10392792766	265168691392

Table A1. *(continued)*

34	390 633 382	−56 527 200 992	1 617 656 173 824
36	1 905 134 695	−308 691 183 938	9 842 665 771 649
38	9 385 453 576	−1 691 769 619 240	59 748 291 677 832
40	46 653 815 395	−9 301 374 102 034	361 933 688 520 940
42	233 788 460 256	−51 286 672 777 080	2 188 328 005 246 304
44	1 180 111 379 105	−283 527 726 282 794	13 208 464 812 265 559
46	5 996 452 414 310	−1 571 151 822 119 216	79 600 379 336 505 560
48	30 653 752 894 948	−8 725 364 469 143 718	479 025 509 574 159 232
50	157 568 531 636 534	−48 552 769 461 088 336	2 878 946 431 929 191 656
52	814 062 277 383 328	−270 670 485 377 401 738	17 281 629 934 637 476 365
54	4 225 485 275 503 702	−1 511 484 024 051 198 680	103 621 922 312 364 296 112
56	22 027 957 435 784 967	−845 3 722 260 102 884 930	620 682 823 263 814 178 484

Table A2. $q = 3$ square lattice Potts model.

	Partition function	Magnetization	Susceptibility
0	1	1	0
1	0	0	0
2	0	0	0
3	0	0	0
4	2	−3	2
5	0	0	0
6	4	−12	16
7	4	−12	16
8	6	−36	100
9	24	−108	216
10	24	−210	844
11	68	−480	1 552
12	190	−1 746	7 844
13	192	−2 340	12 112
14	904	−10 566	60 268
15	1 420	−19 500	118 944
16	3 106	−53 976	424 072
17	9 940	−152 604	1 081 392
18	14 572	−329 424	3 201 728
19	49 268	−971 304	8 670 688
20	102 886	−2 403 291	25 713 154
21	225 004	−5 955 576	67 206 560
22	652 940	−16 858 584	203 077 760
23	1 301 256	−40 337 376	532 881 432
24	3 513 806	−110 301 321	1 558 159 918
25	8 591 792	−287 061 696	4 250 639 632
26	19 326 248	−730 223 208	11 956 293 152
27	52 781 148	−1 985 703 720	33 296 697 848
28	120 709 472	−5 070 001 716	92 820 406 096
29	306 339 824	−13 446 444 720	257 249 275 776
30	779 682 608	−35 650 214 232	721 023 458 656
31	1 852 672 272	−92 442 918 828	1 986 080 278 600
32	4 847 112 666	−247 542 929 499	5 561 045 323 298
33	11 876 028 924	−648 347 258 796	15 359 165 767 512
34	29 820 747 120	−1 713 912 378 552	42 717 426 328 784
35	76 592 341 404	−4 559 593 914 288	118 457 421 095 792
36	189 184 240 720	−11 991 311 519 034	328 170 466 563 836
37	486 960 149 980	−31 943 103 715 128	909 829 346 983 664

Table A2. (continued)

38	1 230 269 248 240	−84 599 939 924 118	252 062 260 622 5 868
39	3 111 387 440 800	−224 265 087 762 144	6 973 368 153 491 880
40	8 008 990 142 050	−597 511 883 594 619	19 322 697 243 220 158
41	20 253 094 484 576	−1 584 231 404 110 704	53 409 977 638 363 032
42	52 022 867 385 004	−4 220 295 103 426 356	147 820 297 067 842 856
43	133 290 716 187 904	−11 234 571 367 790 256	408 655 295 665 071 080
44	340 509 251 651 724	−29 892 611 571 334 848	1 129 521 213 462 962 520
45	878 668 731 837 260	−79 763 126 301 078 204	3 122 011 116 123 891 464
46	2 252 826 675 055 124	−212 500 082 474 434 470	8 624 059 746 484 047 468
47	5 806 881 993 986 032	−567 062 477 783 225 940	23 820 051 913 808 354 000

Table A3. $q = 4$ square lattice Potts model.

	Partition function	Magnetization	Susceptibility
0	1	1	0
1	0	0	0
2	0	0	0
3	0	0	0
4	3	−4	3
5	0	0	0
6	6	−16	24
7	12	−32	48
8	3	−28	120
9	72	−288	648
10	66	−400	1 608
11	144	−1 024	4 176
12	822	−5 268	21 093
13	480	−5 920	38 064
14	3 624	−32 160	175 608
15	8 508	−82 720	494 616
16	10 482	−163 020	1 365 726
17	65 856	−737 568	5 077 200
18	94 794	−1 482 784	13 549 704
19	289 452	−4 644 992	43 359 768
20	1 008 420	−15 095 436	140 590 629
21	1 561 032	−33 307 648	389 348 688
22	6 503 532	−117 747 376	1 296 882 504
23	15 224 016	−312 435 552	3 834 279 072
24	34 976 979	−842 726 356	11 499 126 642
25	125 988 144	−2 747 491 616	36 680 416 368
26	263 308 986	−7 020 371 952	107 193 301 920
27	805 096 764	−21 348 043 296	333 178 056 720
28	2 319 752 694	−62 732 977 996	1 019 415 082 779
29	5 402 283 396	−169 814 283 264	3 037 827 148 632
30	17 415 097 542	−524 175 339 168	9 438 120 599 520
31	44 310 604 860	−1 465 377 774 880	28 340 399 493 144
32	12 026 2 240 257	−4 227 843 277 380	86 034 549 347 280
33	361 259 402 196	−12 642 298 828 704	263 586 587 279 472
34	915 351 056 190	−35 439 363 555 136	791 349 060 376 776
35	2 690 038 490 904	−105 238 706 111 616	2 417 035 981 737 624
36	7 502 832 907 557	−305 746 580 682 940	7 324 176 466 760 445
37	20 120 170 776 144	−877 576 741 412 064	22 116 375 075 991 056

Table A3. (continued)

38	59 297 057 120 916	−2 604 469 134 327 440	67 378 910 515 598 736
39	160 763 118 088 260	−7 498 478 213 381 792	203 361 542 589 030 720
40	453 982 230 850 713	−21 932 701 743 507 244	616 448 061 791 922 708
41	1 302 433 413 699 684	−64 438 624 831 555 744	1 869 466 917 535 240 656
42	3 570 759 048 806 208	−186 505 497 431 699 280	5 644 507 206 242 675 400
43	10 294 158 055 979 004	−549 250 323 031 345 792	17 116 665 818 567 515 608

Table A4. $q = 5$ square lattice Potts model.

	Partition function	Magnetization	Susceptibility
0	1	1	0
1	0	0	0
2	0	0	0
3	0	0	0
4	4	−5	4
5	0	0	0
6	8	−20	32
7	24	−60	96
8	−4	−10	120
9	144	−540	1 296
10	176	−830	3 032
11	168	−1 440	7 392
12	2 348	−12 930	48 856
13	1 200	−12 660	86 496
14	8 792	−72 250	405 368
15	34 056	−266 220	1 473 600
16	21 092	−364 490	3 423 800
17	249 768	−2 407 020	16 113 120
18	466 952	−5 493 880	45 751 360
19	894 840	−14 148 000	141 321 696
20	5 545 356	−66 328 975	563 442 380
21	7 573 416	−133 669 680	1 532 848 896
22	31 825 552	−507 900 420	5 523 490 864
23	109 857 648	−1 701 343 560	18 731 006 352
24	183 834 532	−4 003 389 435	55 057 588 668
25	911 149 824	−16 157 898 840	201 014 668 032
26	2 193 242 320	−44 696 695 560	626 745 614 848
27	5 622 993 528	−131 231 068 680	2 011 978 174 032
28	22 900 219 536	−476 230 251 170	6 990 596 535 720
29	49 840 002 048	−1 272 406 453 680	21 530 011 859 136
30	170 996 310 488	−4 263 523 739 780	72 383 200 786 800
31	547 847 760 000	−13 708 308 759 180	238 951 933 550 064
32	1 328 084 520 588	−38 993 908 456 195	754 059 402 907 804
33	4 859 021 632 872	−133 683 456 488 820	2 546 174 053 634 736
34	13 419 413 642 968	−401 454 864 043 460	8 178 788 230 326 672
35	38 470 066 484 088	−1 235 148 320 717 160	26 521 657 626 107 232
36	131 314 151 349 976	−4 088 698 146 399 590	88 159 797 282 578 792
37	351 804 517 490 808	−12 164 467 726 392 600	282 022 546 158 446 016
38	1 124 247 366 814 936	−39 182 468 233 883 110	927 533 183 411 394 600
39	3 501 627 546 572 496	−124 429 743 346 539 840	3 032 466 102 194 828 496

Table A5. $q = 6$ square lattice Potts model.

	Partition function	Magnetization	Susceptibility
0	1	1	0
1	0	0	0
2	0	0	0
3	0	0	0
4	5	−6	5
5	0	0	0
6	10	−24	40
7	40	−96	160
8	−15	18	100
9	240	−864	2 160
10	390	−1 608	5 440
11	80	−1 536	10 720
12	5 200	−26 478	97 835
13	3 120	−26 208	172 960
14	16 000	−131 472	781 960
15	100 360	−695 136	3 635 280
16	35 830	−705 282	7 301 860
17	676 000	−6 069 600	40 513 440
18	1 760 530	−17 145 840	130 258 520
19	1 919 240	−33 196 224	362 286 640
20	20 588 740	−219 104 358	1 769 583 805
21	31 455 520	−454 883 520	4 908 084 320
22	102 369 320	−1 589 271 912	17 743 365 920
23	534 753 600	−7 015 071 072	71 015 805 120
24	750 136 775	−14 801 255 718	202 223 457 160
25	4 129 522 880	−66 670 704 288	808 396 021 120
26	13 033 586 990	−217 598 415 192	2 788 938 475 160
27	25 901 162 520	−571 905 903 072	8 810 020 871 360
28	139 208 616 380	−2 509 604 877 582	34 575 265 840 675
29	331 986 591 080	−7 057 292 732 736	111 570 542 543 280
30	1 007 597 125 790	−23 164 508 689 248	386 492 454 099 760
31	4 245 621 181 560	−88 855 085 448 096	1 424 843 663 011 600
32	9 716 604 610 065	−248 631 471 063 042	4 599 792 447 092 840
33	37 562 154 772 200	−915 400 373 994 912	16 638 531 775 796 160
34	125 381 210 315 310	−3 099 289 015 999 584	58 025 774 058 205 040
35	327 356 630 630 880	−9 366 267 243 722 112	193 486 559 359 486 160
36	1 307 950 724 960 515	−34 832 974 106 130 846	700 334 584 661 741 705
37	3 808 355 804 296 960	−110 265 822 998 373 408	2 373 578 442 196 407 200
38	11 776 254 793 724 920	−362 663 928 629 146 008	8 176 367 783 196 681 800
39	43 485 616 464 063 400	−1 292 915 644 379 929 056	29 029 320 310 487 709 600

Table A6. $q = 7$ square lattice Potts model.

	Partition function	Magnetization	Susceptibility
0	1	1	0
1	0	0	0
2	0	0	0
3	0	0	0
4	6	−7	6
5	0	0	0
6	12	−28	48

Table A6. (continued)

7	60	−140	240
8	−30	56	60
9	360	−1 260	3 240
10	744	−2 842	9 156
11	−180	−1 120	13 680
12	9 834	−47 754	175 116
13	7 440	−51 940	319 920
14	23 952	−207 102	1 329 156
15	240 180	−1 547 980	7 815 840
16	65 046	−1 295 028	14 115 696
17	1 478 940	−12 844 860	86 758 800
18	5 278 404	−45 718 288	323 797 728
19	3 298 380	−66 714 200	798 016 320
20	59 010 978	−588 028 203	4 634 018 046
21	110 649 780	−1 360 197 160	13 634 236 320
22	253 716 060	−4 028 581 648	47 009 324 592
23	1 946 334 840	−23 146 611 600	221 905 374 120
24	2 709 355 266	−47 264 561 869	625 484 761 002
25	13 823 742 000	−214 532 828 720	2 603 227 959 120
26	58 205 917 656	−851 529 468 216	10 144 975 819 968
27	96 209 517 780	−2 020 426 505 160	31 145 410 317 960
28	603 180 202 992	−10 061 684 149 216	134 214 946 004 952
29	1 719 615 462 960	−31 538 686 100 880	464 836 718 946 240
30	4 359 818 416 824	−96 228 336 284 304	1 603 857 725 501 712
31	23 080 980 564 720	−433 388 885 935 180	6 571 731 067 255 800
32	55 306 170 014 214	−1 246 332 722 187 651	21 917 612 779 277 742
33	201 109 899 966 180	−4 603 433 032 789 500	82 056 490 326 269 640
34	826 406 173 630 152	−17 921 451 461 635 792	313 584 713 235 621 024
35	2 050 391 032 558 740	−53 354 558 614 082 880	1 065 067 369 034 097 360

Table A7. $q = 8$ square lattice Potts model.

	Partition function	Magnetization	Susceptibility
0	1	1	0
1	0	0	0
2	0	0	0
3	0	0	0
4	7	−8	7
5	0	0	0
6	14	−32	56
7	84	−192	336
8	−49	104	0
9	504	−1 728	4 536
10	1 274	−4 640	14 504
11	−672	0	15 792
12	16 730	−78 696	288 169
13	15 792	−96 960	556 752
14	30 464	−293 056	2 062 088
15	497 700	−3 061 440	15 132 264
16	135 506	−2 341 976	25 582 802
17	2 802 240	−24 026 304	165 495 792
18	13 293 602	−106 819 264	720 185 368

Table A7. (continued)

19	5 007 828	−122 471 424	1 588 846 728
20	141 712 956	−1 357 292 536	10 588 862 669
21	330 138 984	−3 611 133 696	33 856 668 720
22	530 780 740	−8 864 133 984	108 773 186 200
23	5 749 786 896	−64 316 947 776	596 266 427 232
24	8 786 258 971	−135 447 363 816	1 709 093 729 238
25	37 764 443 472	−579 146 076 096	7 126 592 218 032
26	207 473 441 098	−2 784 296 946 528	31 462 130 162 512
27	316 373 962 548	−6 222 589 183 680	94 727 643 465 168
28	2 063 246 723 202	−32 959 913 315 576	433 862 737 592 571
29	7 171 329 549 804	−118 042 425 288 960	1 637 231 047 784 136
30	15 593 869 260 998	−333 304 465 512 512	5 555 976 600 192 816
31	96 599 146 653 492	−1 696 556 998 769 088	24 859 503 635 800 680
32	258 118 001 999 361	−5 212 673 379 084 904	86 979 393 078 997 048
33	842 597 112 773 724	−18 588 152 419 905 216	328 938 272 280 954 672
34	4 144 519 614 837 910	−82 387 658 612 630 912	1 373 037 610 865 700 024
35	10 425 182 606 664 504	−248 254 672 744 051 968	4 775 687 621 042 306 472

Table A8. $q = 9$ square lattice Potts model.

	Partition function	Magnetization	Susceptibility
0	1	1	0
1	0	0	0
2	0	0	0
3	0	0	0
4	8	−9	8
5	0	0	0
6	16	−36	64
7	112	−252	448
8	−72	162	−80
9	672	−2 268	6 048
10	2 016	−7 110	21 808
11	−1 456	2 016	16 576
12	26 392	−121 338	444 848
13	30 240	−170 100	919 744
14	32 464	−379 890	2 987 248
15	929 488	−5 535 180	27 006 336
16	297 352	−4 188 870	44 211 856
17	4 789 456	−41 053 068	289 598 400
18	29 374 864	−224 389 944	1 462 179 584
19	7 376 432	−213 735 312	2 947 217 728
20	299 855 320	−2 801 005 551	21 805 200 184
21	855 405 712	−8 621 268 768	76 569 774 848
22	993 563 072	−17 694 788 796	228 101 077 664
23	14 579 082 336	−156 613 057 272	1 422 679 507 872
24	25 470 896 744	−353 319 727 311	4 234 628 561 272
25	89 577 509 504	−1 377 018 275 976	17 279 449 575 808
26	621 970 314 656	−7 875 911 823 192	85 777 377 714 560
27	950 903 499 504	−17 325 165 946 536	257 362 505 607 200
28	5 956 796 021 984	−92 844 355 601 646	1 218 813 810 751 024
29	24 987 280 793 216	−380 787 988 636 944	5 033 799 620 045 952
30	49 220 523 168 272	−1 014 469 914 963 708	16 823 743 332 996 640
31	333 054 308 294 400	−5 604 585 935 358 060	80 496 531 779 269 984

Table A9. $q = 10$ square lattice Potts model.

	Partition function	Magnetization	Susceptibility
0	1	1	0
1	0	0	0
2	0	0	0
3	0	0	0
4	9	−10	9
5	0	0	0
6	18	−40	72
7	144	−320	576
8	−99	230	−180
9	864	−2 880	7 776
10	3 006	−1 036	31 392
11	−2 592	5 120	15 552
12	39 348	−177 810	653 391
13	53 280	−281 920	1 452 096
14	25 992	−454 800	4 102 488
15	1 605 456	−9 336 640	45 188 640
16	631 062	−7 348 830	73 492 200
17	7 576 128	−65 490 240	474 281 280
18	58 741 002	−432 882 160	2 754 369 000
19	11 555 280	−362 940 800	5 188 189 536
20	577 400 796	−5 306 650 050	41 404 785 525
21	1 976 057 856	−18 788 133 760	159 998 233 536
22	1 734 448 752	−32 994 025 240	444 259 126 224
23	32 942 677 248	−344 082 515 520	3 087 795 261 312
24	66 351 732 687	−846 585 038 170	9 670 634 655 948
25	192 109 371 264	−2 979 703 762 880	38 172 808 778 112
26	1 631 012 967 630	−19 830 863 413 320	210 547 287 205 128
27	2 631 716 943 408	−44 403 893 943 360	639 171 550 340 352
28	15 171 912 275 256	−233 105 484 042 490	3 068 237 140 062 435
29	75 277 805 632 848	−1 085 356 198 700 160	13 833 211 331 827 296
30	141 748 393 419 918	−2 799 174 780 003 360	45 883 584 616 400 640
31	991 716 118 694 640	−16 218 543 098 061 760	230 335 082 028 303 264

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